Multivariable Calculus MATH20901, Exam Solutions, January 2017

1. (a) *(From first part of course, revision of Calculus 1, trivial unseen example)*

   (i) With $F_1 = x^2 + y^3$, $F_2 = \cos x + \sin y$

$$\mathbf{F}'(\mathbf{x}) = \begin{pmatrix} \partial F_1/\partial x & \partial F_1/\partial y \\ \partial F_2/\partial x & \partial F_2/\partial y \end{pmatrix} = \begin{pmatrix} 2x & 3y^2 \\ -\sin x & \cos y \end{pmatrix}$$

   (ii) *(Following methods in notes, worksheets)*
   From Taylor series around $\mathbf{x}_0 = (x_0, y_0) = (\frac{1}{2}\pi, 0)$

$$F_1(\mathbf{x}) = F_1(\mathbf{x}_0) + (x - \tfrac{1}{2}\pi)\frac{\partial F_1}{\partial x}(\mathbf{x}_0) + y\frac{\partial F_1}{\partial y}(\mathbf{x}_0) + h.o.t. \approx \tfrac{1}{4}\pi^2 + (x - \tfrac{1}{2}\pi)\pi$$

   and

$$F_2(\mathbf{x}) = F_2(\mathbf{x}_0) + (x - \tfrac{1}{2}\pi)\frac{\partial F_2}{\partial x}(\mathbf{x}_0) + y\frac{\partial F_2}{\partial y}(\mathbf{x}_0) + h.o.t. \approx -(x - \tfrac{1}{2}\pi) + y$$

   gives answer.

   (iii) *(Slightly harder part of course material. Similar examples on worksheets.)*
   The system

$$u = x^2 + y^3, \qquad v = \cos x + \sin y$$

   is invertible to get $x = x(u, v)$ and $y = y(u, v)$ provided the Jacobian $J_{\mathbf{F}} \neq 0$. Here

$$J_{\mathbf{F}} = 2x\cos y + 3y^2 \sin x.$$

   Along $x_0 = 0$ $J_{\mathbf{F}} = 0$ so not invertible. Along $x_0 = \frac{1}{2}\pi$ $J_{\mathbf{F}} = \pi\cos y + 3y^2$. This is always positive as a quick sketch will confirm. Since it doesn't vanish, the system is invertible along this line.

(b) (i) (*First part tests path integral calculation (third part of course). V. similar to notes/examples*)
   Parametrise circle with $\mathbf{r} = \mathbf{p}(\theta) = (a\cos\theta, a\sin\theta)$ for $0 < \theta \leq 2\pi$. Then $\mathbf{p}'(\theta) = (-a\sin\theta, a\cos\theta)$ and so the integral is

$$\int_0^{2\pi} \left(\frac{-\sin\theta}{a}, \frac{\cos\theta}{a}\right) \cdot (-a\sin\theta, a\cos\theta)\, d\theta = 2\pi.$$

   (ii) (*One or two examples on worksheets to follow – a less well trodden part of the course*)
   Green's theorem in the plane states that if $\mathbf{G} = (P, Q)$ then

$$\int_C \mathbf{G} \cdot d\mathbf{r} = \int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

   where S is the interior of C. Here we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}$$

   and so

$$\int_C \mathbf{G} \cdot d\mathbf{r} = \int_S 0\, dxdy = 0.$$

   (iii) (*Unseen. Similar issue referenced in one question on a problem sheet*)
   They do not agree, but normally they should... the problem here is that P and Q are singular at the origin and non-integrable and Green's theorem in the plane (Stokes' theorem) assumes integrability.

2. *Tests differential calculus (2nd part of course) and integral calculus (3rd part of course).*

   (a) *(Bookwork)*
   (i) $\nabla \cdot \mathbf{r} = \partial x_i / \partial x_i = 3$;
   (ii) $\nabla \sqrt{x^2+y^2+z^2} = (x,y,z)/\sqrt{x^2+y^2+z^2} = \mathbf{r}/r$.

   (b) (i) *(Homework example)*

$$\nabla \cdot (f \nabla g) = \frac{\partial}{\partial x_i}\left(f \frac{\partial g}{\partial x_i} \right) = \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i} + f \frac{\partial^2 g}{\partial x_i^2} = \nabla f \cdot \nabla g + f \Delta g$$

   (ii) *(Bookwork)*
   By Divergence theorem and using part (i)

$$\int_S \hat{\mathbf{n}} \cdot (f \nabla g)\, dS = \int_V \nabla \cdot (f \nabla g)\, dV = \int_V \nabla f \cdot \nabla g + f \Delta g \, dV$$

   and then repeat with f and g interchanged and subtract to get result.

   (c) *(Unseen example, mixture of easier and harder bits.)*
   In notes we have $\nabla f(r) = f'(r) \nabla r = f'(r)\mathbf{r}/r$ from chain rule and part (a)(ii). So for $r < a$, $\mathbf{F} = \nabla f = -\mathbf{r}/a^3$ and for $r > a$, $\mathbf{F} = \nabla f = -\mathbf{r}/r^3$. So cts across $r = a$. Then for $r < a$

$$\Delta f = (-1/a^3) \nabla \cdot \mathbf{r} = -3/a^3$$

   whilst for $r > a$, using part (b)(i)

$$\Delta f = (-1/r^3)\nabla \cdot \mathbf{r} - \nabla(1/r^3) \cdot \mathbf{r} = -3/r^3 + 3/r^5 \mathbf{r} \cdot \mathbf{r} = 0$$

   using part (a) and $\mathbf{r} \cdot \mathbf{r} = r^2$. So as required, with $C = -3/a^3$.

   (d) *(Unseen, intended to be more challenging)*
   With $\Delta g = 0$, choose f from part (c) so that $\Delta f = -3/a^3$. Now $f = 0$ on $r = a$ and $\hat{\mathbf{n}} \cdot \nabla f = (\mathbf{r}/a) \cdot (-\mathbf{r}/a^3)$ on $r = a$ which equates to $-1/a^2$ on $r = a$. Putting f and g into Green's Identity (part (b)(ii)) gives

$$(-3/a^3) \int_V g \, dV = (-1/a^2) \int_{\partial V} g \, dS$$

   and hence result.
   If $g = 1$, then $\Delta g = 0$ and the volume integral is just $\frac{4}{3}\pi a^3$ whilst the surface integral is $4\pi a^2$ and hence the formula works.