This paper contains four questions
All answers will be used for assessment.
Calculators are not permitted in this examination.
1. In an experiment, a sphere of radius $a$ is heated uniformly to a temperature of $100^\circ C$ before being dunked in cold water. The temperature inside a sphere is determined by the solution, $\theta(r, t)$ of heat equation in spherical polars,

$$\frac{\partial \theta}{\partial t} = D \frac{\partial}{\partial r} \left( r^2 \frac{\partial \theta}{\partial r} \right), \quad 0 < r < a, \quad t > 0. \quad (1)$$

where $D$ is a diffusion coefficient with

$$\theta(a, t) = 0, \quad \text{for } t > 0 \quad \text{and} \quad \theta(r, 0) = 100, \quad \text{for } 0 < r < a.$$

(a) (6 marks)

Make the substitution $\theta(r, t) = u(r, t)/r$ into (1) to show that $u(r, t)$ satisfies

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial r^2}$$

(b) (2 marks)

By referring to the substitution in part (a), explain why the boundary condition on the function $u$ at $r = 0$ must be $u(0, t) = 0$ ?

(c) (8 marks)

Use separation of variables to show that a general solution satisfying the heat equation and boundary conditions can be expressed as

$$u(r, t) = \sum_{n=1}^{\infty} a_n e^{-\mu_n^2 D t} \sin(\mu_n r)$$

where $a_n$ are coefficients and the values of $\mu_n$ should be identified.

(d) (6 marks)

Now apply the initial condition to find $a_n$.

(e) (3 marks)

Determine the dominant large-time behavior of $\theta(r, t)$.
2. (a) (5 marks)
Consider string vibrating in the air with air resistance proportional to the velocity of the string. Starting from first principles, derive the one dimensional equation for the displacement, \( u(x,t) \), on a string of constant tension \( T \) and mass per unit length \( \rho \).

\[
\ddot{u} + \alpha \dot{u} = c^2 \dddot{u}
\]  

(2)

and where \( c \) should be defined in terms of \( T \) and \( \rho \) and \( \alpha \) is a constant corresponding to the air resistance.

(b) (3 marks)
Show that \( v(x,t) = f(x-ct) + g(x+ct) \) for arbitrary functions \( f \) and \( g \) satisfies the wave equation

\[
v_{tt} = c^2 v_{xx}, \quad t > 0, \quad -\infty < x < \infty
\]

(3)

(c) (6 marks)
Derive the D’Alembert’s solution for the wave equation (3) for a function \( v(x,t) \), subject to the initial conditions

\[
v(x,0) = \phi(x), \quad v_t(x,0) = \psi(x).
\]

(d) (8 marks)
Using Duhamel’s principle derive the solution of the following problem

\[
w_{tt} = c^2 w_{xx} + f(x,t), \quad t > 0, \quad -\infty < x < \infty
\]

subject to the initial data

\[
w(x,0) = 0, \quad w_t(x,0) = 0.
\]

(e) (3 marks)
Show that \( u(x,t) \) solving the following problem

\[
u_{tt} = c^2 u_{xx} + f(x,t), \quad t > 0, \quad -\infty < x < \infty
\]

subject to the initial data

\[
u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x).
\]

can be represented as \( u(x,t) = v(x,t) + w(x,t) \), where \( v \) and \( w \) are defined in (c) and (d), respectively.
3. Consider the following first order non-linear PDE for the function \( u(x,t) \):

\[
u_t + g(u)u_x = 0,
\]

where \( g \) is a given function of \( u \), with initial condition

\[
u(x,0) = f(x) \equiv \begin{cases} 
0, & |x| > 1, \\
1 - |x|, & |x| < 1.
\end{cases}
\]

(a) (7 marks)
Show that the characteristic curves, along which the solution is constant, are defined by \( \xi = \text{constant} \) where \( x = \xi + g(f(\xi))t \).

(b) (2 marks)
First, take \( g(u) = c \), a constant. Sketch the characteristic curves and (on a separate graph) the solution at \( t = 0, t = 1/c, t = 2/c \).

(c) (8 marks)
For the remainder of the question take \( g(u) = 1 + u \). Find characteristic curves \( x(t) \) and solution \( u(x,t) \).

(d) (5 marks)
State which property of the characteristic curves can be identified with a shock. Sketch characteristic curves and find the time \( t_s \) and point in space \( x_s \) at which the solution first shocks.

(e) (3 marks)
Sketch roughly how you expect the solution to develop in time, up to the point at which it shocks.
4. In plane polar coordinates, \((r, \theta)\), the steady-state diffusion of heat \(u(r, \theta)\) in the cross-section of a long cylinder of radius \(a\) is given by Laplace’s equation,

\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < r < a.
\]

(a) (8 marks) Separate variables, to show that the \(\theta\) variation can be represented by the function \(\Theta(\theta) = A_m \cos m\theta + B_m \sin m\theta\) for arbitrary coefficients \(A_m\) and \(B_m\). What values of \(m\) should be taken and why? Also derive and solve the ODE for the radial variation \(R(r)\), of \(u\).

(b) (4 marks) On \(r = a\), the temperature is a given function, \(u(a, \theta) = f(\theta)\) where \(f(\theta) = f(-\theta)\). Explain why the general solution should be expressed as

\[
u(r, \theta) = \sum_{m=0}^{\infty} c_m r^m \cos m\theta
\]

and provide the expression for coefficients \(c_m\).

(c) (8 marks)
Show that if \(u\) is harmonic function in \(\Omega = \{(x, y) : x^2 + y^2 \leq 1\}\) (i.e. \(u \in C^2(\Omega)\) and \(\Delta u = 0\) in \(\Omega\)) then the maximum and minimum of \(u\) is achieved on the boundary \(\partial\Omega\).

(d) (5 marks)
Find the solution of the following problem

\[
\Delta u = (x^2 + y^2)^4 \quad \text{in } \Omega = \{(x, y) : x^2 + y^2 \leq 1\}, \quad u(x, y) = 1 \quad \text{on } \partial\Omega.
\]

End of examination.